PYTHAGOREAN HARMONY MATHEMATICS AND BUILDING TECHNIQUES: THE SECOND TEMPLE OF HERA ('POSEIDON') AT PAESTUM

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It can be demonstrated that the architect of the second temple of Hera at Paestum (ca. 460 BC) built in one Pythagorean (3, 4) rectangle, two squares, two concentric circles, one golden isosceles triangle, two golden right-angled triangles and one hexagon by the ratios 2:5, 3:5, 4:5, 5:6, 5:7 and 5:8. Apart from one horizontal rectangular plane (the ratio 2:5 applies to its sides) that does not correspond factually to set distances, all vertical figures can be measured, but no figure can be perceived by men as it is all virtual reality. Looking at the front two opposite ideas come to mind. Were these imperceptible ratios intended as means of defence to counteract demonic activity or as creation in honour of the gods? The architect was familiar with the gnomon, that links Indian and Pythagorean geometry. The utility of methodical shifting of columns out of their ideal aesthetic position is illustrated.

Introduction

In archaic and classical Greece, Pythagorean harmony was just adaptation of different things to each other in order to settle planned totality (Naredi-Rainer 1982, 11-15). Herakleitos of Ephesos (about 500 BC), a non-Pythagorean philosopher, says that things are connected via oppositions, and this produces harmony (Diels/Kranz 1956, I 152, Herakleitos frg. B 8). Pythagorean ontology first occurs in Philolaos (about 400 BC). This philosopher assigned the triangle to male and the square to female deities (Diels/Kranz 1956, I 402, Philolaos frg. A 14). According to Aristotle's report (*Metaphys*. A5, 986^{a} 24-25) the Pythagoreans held that odd numbers are male, even numbers female. Yet what is actually meant by such statements? Raglan (1949, 111) says: "The attribution of sex to inanimate objects – stars, rivers, boats, and so on – cannot be the result of observation, even faulty observation, and can never have served any useful purpose". One may then suggest that Raglan's statement includes geometrical figures and numbers too. The close connection of philosophy, theology and mathematics is difficult to grasp, but generally felt to be correct by intuition. Unfortunately, this gives rise to manifold interpretations resulting in disorder in modern research.

Perhaps, the problem is not altogether hopeless. Kuznetsova (2005) states that harmony derives origin from ancient mathematical theories. Temples in South Italy, the nucleus of Pythagorean activity, are obvious places to look for the evidence. As mathematics is an exact science, it is sufficient to find an example by which correct judgement can be made. Of course, the criterion is the almost mathematical accuracy with which the theoretical size of the figures expressed in specific units (Attic feet) agree with those actually measured on the temple (centimetres converted into Attic feet). The fixed foot length (32.66 cm) is no matter for argument as the architect used telling numbers (618 and 1000) in his design of the golden isosceles triangles: the explanation of the actual measurement in telling numbers will obviously only work for a specific value of the foot length in the metric system (1 foot = 16 dactyls, 618 dactyls = $38^5/_8$ feet and 1000 dactyls = $62^1/_2$ feet). The result of this study is so arranged that it can easily be read by non-specialists in the field of Greek architecture.

Towards unravelling Greek temple design

In my paper on the temple of Athena at Paestum I disputed the current opinion among modern investigators of Greek architecture that the evolution of an aesthetically satisfying building was the major preoccupation of the temple architect (de Zwarte 2006). I presented strong evidence that magic protection of temples overrules aesthetic matters. This is not to say that the aesthetical appearance was neglected by the ancient architect, but only that aesthetical matters are secondary to the screening of the temple from evil spirits. I certainly do not exclude the possibility that my own explanation of the facts needs to be modified on the strength of new evidence or by a different interpretation of the existing evidence by scholars with expert knowledge of the written philosophical sources. However, I hold firmly that the idea of a Greek architect being *only* at work to realize the best aesthetic appearance of a temple can no longer be maintained.

The second temple of Hera at Paestum supplies two new points in support of our thesis. The first one is immediately clear: A virtual horizontal rectangular plane *without any physical relation to the temple* can never be explained as an aesthetic feature. The second one, the relation between the triglyph – an ornament in the frieze of a Doric temple, repeated at equal intervals – and the column, requires an explanation.

At this point a brief description of the temple may be convenient (Fig. 1). On the upper step (stylobate) of the platform stand 6 x 14 columns, the corner columns twice counted. The columns, included capital and plate (abacus), support the superstructure. The superstructure is divided into three parts: (1) The supporting member, carried from column to column (architrave). The projecting fillet (taenia) which crowns the architrave plays, rather unexpected, a part in this study. (2) The decorative portion (frieze). The architrave (without taenia) and the frieze share the same dimensions in their horizontal plan. (3) The crowning and projecting member (horizontal *geison*). On the fronts the triangular wall (*tympanon*)

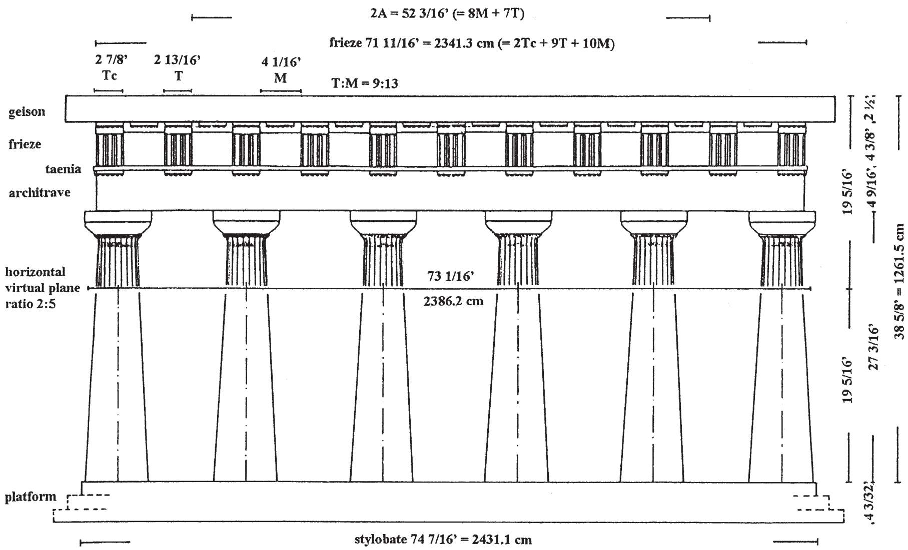

Fig. 1. The front of the second temple of Hera ('Poseidon') at Paestum.

and the raking *geison* are still extant on top of the horizontal *geison*, but not indicated in Fig. 1 as no measurement is available. A publication in monograph form is still missing, so I could only use the summary description by Krauss (1941) and the additional information which Mertens (1984) gives in the appendix to his monograph on the temple of Segesta.

Let us return to the frieze. The constituent parts are identical on all sides of the temple. The frieze is also regular, which means that the triglyphs are of equal width (T = $2^{13}/_{16}$') with the exception of the corner triglyphs (Tc = $2^{7}/_{8}$'). The spaces (metopes) between the triglyphs are equal everywhere (M = $4^{1}/_{16}$').
The ideal aesthetical position of the triglyph in relation to the column is centred over the column axis. In Doric temple design, the corner triglyph is an exception to this rule. However, the architect of our temple made the axial intercolumniation at the corners on the front smaller than the other ones (single angle contraction) and on the flank the same applies to this spacing, but also to the axial intercolumniation next to it (double angle contraction). Consequently, counted from the corners of the temple, the ideal aesthetical position of the second column on front and the second and third column on flank could not be maintained as the architect preferred regularity in the frieze instead of altering the width of metopes to compensate the column-shift. Actually, these columns are shifted with regard to the axis of the triglyph as a consequence of the angle contraction. But what

about the others? Krauss[1] took the three remaining axial intercolumniations on the front and nine on the flank for normal, that is of equal width. Strangely enough, he also said that more (or all?) columns are shifted. It is here that the absence of a monograph is most deeply felt. Actually he said that the triglyphs are shifted, but that is impossible as the shifting of one triglyph causes the shifting of the whole frieze. Later on he stated that the architect's design started from the superstructure to settle conclusively the position of the columns[2]. I attach importance to the fact that Krauss did not make an appeal to error in execution. He was ready to accept that a series of equal axial intercolumniations is compatible with the shifting of columns.

Krauss was certainly right when he attached much importance to the frieze. However, it can be proved that the architect started his design from the bottom of the platform, resulting in a temple in which 3 axial intercolumniations on the front are of equal width, just as nine on the flank. Consequently, the columns with triglyphs to match are centred. This design was aesthetically perfect in every respect and displays already several interesting features. However, the architect wanted more and was ready to achieve it at the sacrifice of the perfect aesthetical appearance. In the final design more columns were systematically shifted – the relevant axial intercolumniations vary in width accordingly – and the explanation of this is the insertion of manifold Pythagorean mathematics in the design of the temple. In fact, the temple design is subordinated to the virtual mathematical concept.

Virtuality in Greek temple architecture

The existence of virtual numbers and mathematical figures deserves careful attention as it brings on a radical change in modern thinking on Greek architecture, mathematics and philosophy. The rapid progress in this field of research in recent years is the result of the discovery that the Greeks used two foot lengths of *standardized* length, the Ionic foot (IF) of 29.86 cm (divided into 16 dactyls) and a derivative of this foot ($17^1/_2$ Ionic dactyls), the Attic foot (AF) of 32.66 cm (divided into 16 dactyls) (de Zwarte 1994 and 2006, Digression).

To my best knowledge, the identification of 'virtual reality' is brand-new and the research is still in its infancy. A virtual number is the result of the system of proportion of the type let x = y – z, in which y and z represent length and width of a rectangle, expressed in feet and identical fractions of feet. Consequently, x is a

[1] Krauss 1976, 50-53. This is the unaltered third edition of the original text printed in 1941, but it includes a foreword by Gruben and an appendix by Mertens in which, however, information on measurements is lacking.

[2] Krauss 1976, 62: "Denn da der Fries, von diesen Verschiebungen abgesehen, den Grundriss in einfacher Form enthällt, während in der Säulenstellung die Regelmässigkeit mit der Eckkontraktion aufgegeben werden musste, ist beim Poseidontempel der Entwurf folgerichtig vom Gebälk ausgegangen."

virtual whole number of feet. Up to now x is always a number that can be divided by 5, but that might be a coincidence as the evidence is still meagre, including fresh evidence, presented in Table 1[3].

Table 1. Rectangles (* = virtual): the difference of length and width is a virtual number

Place	Rectangle	measured (cm)	Interpretation cm feet
Athens,	Stylobate	L – W = 3733.3	3732.5 = 125 IF = $197^{3}/_{4}$ – $72^{3}/_{4}$
Parthenon,	Axial, columns*	L – W = 3732.2	3732.5 = 125 IF = $191^{11}/_{16}$ – $66^{11}/_{16}$
Cella	Elevation, front*	W – H = 746.3	746.5 = 25 IF = $66^{11}/_{16}$ – $41^{11}/_{16}$
	Elevation, flank*	L – H = 4478.5	4479.0 = 150 IF = $191^{11}/_{16}$ – $41^{11}/_{16}$
Paestum, Hera I	Outlying altar, platform	L – W = 1493.0	1493.0 = 50 IF = $70^{5}/_{16}$ – $20^{5}/_{16}$
Paestum, Hera II	Cella socle	L – W = 3267.9/ 3268.7	3266.0 = 100 AF = $141^{5}/_{16}$ – $41^{5}/_{16}$

The dimensions of the rectangles in the Parthenon and the Hera II temple (Fig. 2) are difficult to assess because they are surrounded by the colonnade. However, the burnt-offering altar of the Hera I temple is situated in the open. Seen from an angle, from a distance or close at hand, a visitor – whether with or without knowledge of the architect's foot-standard – sees nothing more than two sides of the altar platform, of which the dimensions can be guessed more or less accurately. Only accurate mensuration followed by mental arithmetic reveals the meaningful round number of feet which is the difference of length and width, provided that the foot length is known. Thus such a number is virtual to anyone and a contemporary visiting architect who has the number by hearsay, cannot be sure of it just on basis of assessing the dimensions of the altar by eye.

This altar was built during the lifetime of Pythagoras. Thus it is not really surprising to see that the diagonal is a whole number (Fig. 2). The constructions of the Pythagoreans are often based on near-triples. Whether the Pythagoreans knew that the triple (325, 1125, 1171) is incorrect is a matter of opinion. If they obtained the number 1171 by algebra, they knew it, but if found by geometry, I cannot judge that as the difference between the theoretical length of the diagonal and the practical length is less than 1/10th of a millimetre. In the Hera I temple there is also a virtual square that shows the square root of 2 by the rational approximation 437/309. The construction of the virtual golden isosceles triangle in both temples of Hera at Paestum is also based on a near-triple (Fig. 3). Surely, the accuracy

[3] Athens, Parthenon: de Zwarte 2002, 14-16 and fig. 6, after Mertens 1984 (Parthenon), 66-67; Paestum, Hera I: de Zwarte 2002, 13-14, after Mertens 1993, 3, fig. 2; Paestum, Hera II, after Krauss 1976, 46, fig. 4.

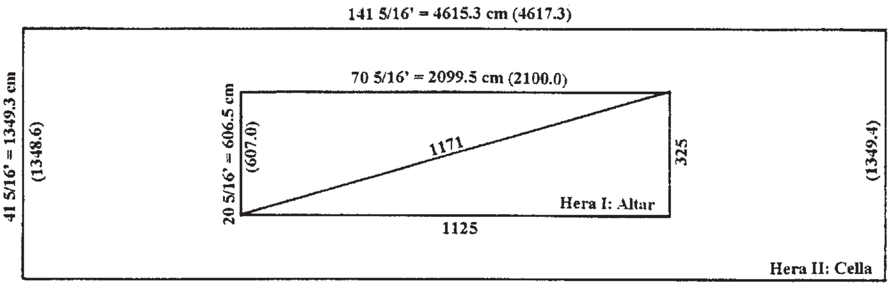

Fig. 2. Temples at Paestum. Rectangles representing the outline of the altar of the first temple of Hera (length minus width is 50 Ionic feet) and the cella of the second temple of Hera (length minus width is 100 Attic feet); measurements between brackets.

with which these figures have been set out in the Hera I temple is asthonishing[4]. Unfortunately, nothing is known in detail about the measuring instruments used by the Greeks, for none have survived.

Virtual figures cannot be seen at a glance (see Table 1: *), as the outline does not coincide with the lines of a single part of the design. For example, the rectangle forming a triglyph is real, but the axis of the triglyph may be a part of a virtual figure. The exception to this rule is the straight line. The figure is virtual if the position of some points is defined by the architect to embody a special meaning. The only example of this sort is extant on the steps of the Hera I temple at Paestum (*ca.* 530 BC). The geometrical progression of Pythagoras – 1, 3, 4, 7, 11, 18, .. – found anew by Lucas (1877), was used in setting out the golden section with the help of the numbers 521, 843 and 1364. The golden section can be expressed as the ratio of two successive integers in this progression[5]. Joints between the stone blocks give the position of the points concerned. Certainly, this was not in the least an easy problem to deal with[6]. Virtual figures are invisible, but

[4] Measurement by Mertens 1993; appended drawing 2 (ground-plan, in temple axis): 341.1 + 812.5 = 1153.6 cm (side of square), 1153.6 + 621.3 = 1774.9 cm (perpendicular); appended drawing 8 (d: inner view): 1153.5 cm (side of square and base of golden triangle); $38^5/_8$ IF = 1153.3 cm and $59^7/_{16}$ IF = 1774.8 cm.

[5] In fact, the golden section is only accurate if geometrically constructed (see Naredi-Rainer 1982, 195), but successive *high* numbers of geometrical progressions give an accurate approximation (Numbers of Fibonacci 89 and 144 give 0.61805.., which is accurate, unlike 5 and 8 of the same series, which give 0.625).

[6] de Zwarte 2002, 11-12. The discovery was the result of a coincidence. In his study on the Hera I temple at Paestum de Waele (1995, 513-518) suggested that the over-all length of a series of stone blocks could be significant to Greek architects in designing the tripartite division of the ground-plan of a temple. Of course, only a modern publication in monograph

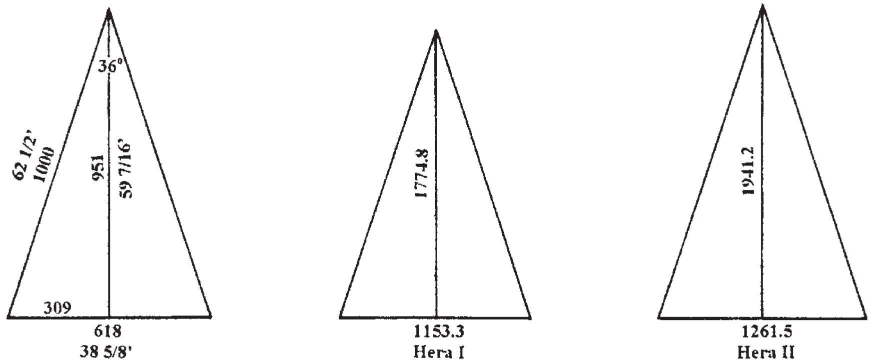

Fig. 3. Temples at Paestum. Telling numbers of the golden isosceles triangle and its metric equivalent in cm if executed in Ionic feet (Hera I) or Attic feet (Hera II).

they can be measured and in this manner their existence can be revealed. This applies to virtual figures embedded in the building. However, the second temple of Hera at Paestum has a novelty, that is a virtual rectangular plane that is not framed in the construction of the temple. So the problem arises how it can be traced. This is a question I will explain below.

Hera temple II: floating rectangular plane

Krauss[7] says (1941): "… in der vorspringenden Taenia des Architravs gemessen, also an der Stelle von dessen grösster Ausladung das klare Seitenverhältnis von $1:2^{1}/_{2}$ …" Unfortunately, he stated only the measurement on the front – 2356 cm; 2356 x $2^{1}/_{2}$ = 5890 cm – so nobody could check whether the ratio is accurate or only approximate, which could give reason to dismiss the claim. There are also objections on systematic grounds (Fig. 1) as Coulton (1975, 62) commented: "It is most unlikely that the architect of the second temple of Hera at Paestum, building a highly conventional temple, was the only man to design a temple so that there was a simple proportion on length to width at the level of the architrave tae-

form (Mertens 1993) – in which every single stone is accurately measured – is suitable to put the idea to the test. Obviously this is also a job for a computer, but an adequate calculating program was lacking. So I spent hours at systematic calculation. The results were ambiguous. Surprisingly, I found connected measures on the middle step on north side, starting from the west edge of the step (1943.5 + 3144.5 = 5088.0 cm) and in identical position on the south side. Within the context of temple design these figures were meaningless, but playing with my calculator the architect's intention became clear. In retrospect I ask myself whether a computer is really helpful in tracing *unexpected* things. However, almost certainly I should also have failed in tracing the golden section if executed only once. On the meaning of the virtual figures in the first temple of Hera at Paestum see de Zwarte 2005.

[7] Krauss 1976, 62-63 (60: 2356 cm measured on front).

nia.” Coulton concluded that this proportion was simply a matter of coincidence.

Table 2. The second temple of Hera (‘Poseidon’) at Paestum			
First published by Krauss in 1941 (K) or by Mertens in 1984 (M)		Measured (cm)	Interpretation (1’ = 32.66 cm)
Stylobate length on front (w/e)	K46	2429.6/2431.6	2431.1 = $74^{7}/_{16}$
Stylobate length on flank (s/n)	K46	5989.1/5996.0	5997.2 = $183^{5}/_{8}$
Corner triglyph width (Tc)	K50	93.8	93.9 = $2^{7}/_{8}$
Triglyph width (T)	M217	91.8	91.9 = $2^{13}/_{16}$
Metope width (M)	M217	132.5	132.7 = $4^{1}/_{16}$
Frieze length on front = 2Tc+9T+10M	calc.	2338.8	2341.3 = $71^{11}/_{16}$
Frieze length on flank = 2Tc+25T+26M	calc.	5927.6	5933.9 = $181^{11}/_{16}$
Geison height	K53	81.0	81.7 = $2^{1}/_{2}$
Frieze height	K50	143.3	142.9 = $4^{3}/_{8}$
Architrave height	K50	148.8	149.0 = $4^{9}/_{16}$
Column height	K49	888.0	887.9 = $27^{3}/_{16}$
Subtotal height	calc.	1261.1	1261.5 = $38^{5}/_{8}$
Platform height	M215	133.5	133.7 = $4^{3}/_{32}$
Total height	calc.	1394.6	1395.2 = $42^{23}/_{32}$

In 1984, Mertens gives additional information (see Table 2), and this permits the calculation of the frieze lengths. As the architrave and frieze lengths are equal, the length of two times the protruding taenia is about 2356 – 2339 = 17 cm. This amount added to the frieze/architrave length on the flank gives about 5945 cm, or 55 cm longer than the ratio 1:$2^{1}/_{2}$ predicted. It will be remembered that Krauss was an outstanding historian of Greek architecture, not a charlatan. As such a difference is extreme, it is not possible to decide on this by an off-hand judgement, but only by plausible conjecture how discoveries of that kind must surely have been made. In fact it is very simple. The ratio in the taenia is a makeshift. The ratio is absolutely correct, but not positioned in the taenia. Krauss’ only hope was that his hint would be of service to other investigators who are better equipped than Krauss himself to make use of it. Indeed, it is not surprising that Krauss could not cope with virtuality, as this is incompatible with his position that temple design is a matter of pure aesthetics. This is still the position almost generally held so far by historians of Greek architecture.

Until recently investigators were in the dark concerning Greek standards of length. Previous work in examining design methods had usually been based on a detailed study of proportion, which can be done without previous knowledge of the foot-standard used by the architect. The first step in analysing a Greek temple in this way is obviously the calculation of the proportion of length to width of the main parts in the simplest terms. In this manner I tread certainly in Krauss’ steps. The ratio of length to width of the stylobate, averaging the 4 possible combinations of the measurement (see Table 2), is 2.465:1; the ratio of the averaged frieze dimensions is 2.534:1; the sum of these ratios is 4.999:2, or almost exactly 5:2. This cannot be accidental. Following Pythagorean practise, I do not write the ratio as $2^{1}/_{2}$:1.

No doubt Krauss soon realized that his discovery was problematic. It is easy to find that the sides of this rectangle are calculable from half the sum of stylobate and frieze dimensions, respectively on the front and on the flank: $^1/_2(74^7/_{16}$' + $71^{11}/_{16}$') : $^1/_2(183^5/_8$' + $181^{11}/_{16}$') = $73^1/_{16}$' (2386.2 cm) : $182^{21}/_{32}$' (5965.6 cm) = 2:5. Such a rectangle cannot be inferred from the building's singular dimensions, and a virtual rectangle did not fit in with Krauss' way of thinking on Greek temple design. As we shall see later on, the dimensions of the plane are fixed this way for the development of the design. At this stage of our explanation I emphasize the fact that the position of the plane is of vital importance. As Krauss had no knowledge of the foot length, he was unable to insert the plane in its correct position, even if he had accepted a virtual position. Fig. 3 gives us the clue. Using telling numbers, the base of the golden isosceles triangle is 618 in number (dactyls), in feet $38^5/_8$', that is 1261.5 cm if the foot length is 32.66 cm; modern measurement gives 1261.1 cm (Table 2). This is the distance between the top of the stylobate and the top of the horizontal *geison* (Fig.1). As the base of the isosceles triangle is standing upright, it follows that the position of the 'perpendicular' is horizontal in this case. The virtual rectangular plane was so placed that the 'perpendicular' of the golden isosceles triangle and the virtual horizontal plane coincide. Thus, the horizontal 'perpendicular' we expect to find is 951 in number (dactyls), in feet $59^7/_{16}$' and converted into the metric system 1941.2 cm. In fact, however, there are two horizontal 'perpendiculars', as the architect planned these isosceles triangles at both corners of the front.
It is here that a remark on the contents of Table 2 may be useful. The height of the platform was originally valued by me at 3 x $1^3/_8$' = $4^1/_8$' = 134.7 cm on the assumption that the steps are of equal height. In this case the resulting total height is $42^3/_4$ feet, which seems more likely in planning the over-all dimension. However, in the course of further study a measure of $42^{23}/_{32}$' emerged level and this time certainly irreplaceable by $42^3/_4$ feet. It follows, if I am on the right track, that the architect had a clear picture of a square with side $42^{23}/_{32}$ feet in his mind.

This rather long digression was needed, I think, to make clear why our knowledge of Greek temple design is still defective in spite of age-long research. Besides, it seems unwise to underrate the intelligence of ancient architects. I fully disagree with Coulton (1974, 86) when he says about the ancient Greek architect: "Such a man is unlikely to have been aware of the need for detailed planning or in the possession of the intellectual concepts that would make it possible". I suggest a new approach for studying Greek architecture, that is free from such erroneous axioms. It is not surprising that historians of mathematics tend to neglect books and articles on Greek architecture as many of these studies indeed are paradoxical and of little advantage to the study of early Greek mathematics. On the other hand, it surprises me that the relation between philosophy and mathematics is still treated in a stepmotherly fashion, as Zhmud (2006, 4) very recently observed that science is largely ignored by modern scholars studying Pythagoras and early Pythagoreanism. However, things will take a turn. The undeniable evidence, that

Pythagorean mathematics was in process of development in the archaic and classical period, is available, but such an evolution can hardly have been limited to Paestum. It requires joint efforts of many students to put other cities and temple-sites on the map.

The front of the Hera temple II

Let us look more closely at the details on the front of the temple (Fig. 4). Because a monograph on this temple is lacking, I have to work with the minimum of evidence based on modern measurements, but I add to Table 3 (see p. 12) as fully and clearly as possible the calculations which complete the evidence, needed to reconstruct the design.

The preliminary plan (Fig. 4a) has single angle contraction and it is aesthetically perfect as the axis of the columns 2 and 3 is in line with the axis of a triglyph, that is, as historians of Greek architecture would like to have it. In this case, however, it does not display the final stage of the plan, our Pythagorean architect must have had in mind. The plan still misses the final touch (see *infra*), but, as to mathematics, the ancient architect already had achieved an impressive result:

1. Two Pythagorean (3, 4) rectangles situated at the corners. The horizontal short side is the distance from the edge of the frieze to the axis of the near central column 3 ($28^{31}/_{32}$ feet). The long side runs from the top of the stylobate to the top of the *geison* ($38^{5}/_{8}$’).
2. Two squares with side $42^{23}/_{32}$ feet, also at the corners. The horizontal side is the distance from the edge of the frieze to the axis of the far central column 3. The vertical side is the total height already discussed above.
3. An identically equal square in central position. The horizontal side is given by the distance between the intersections of the diagonals of the (3, 4) rectangles ($1/2 \times 28^{31}/_{32}$’ + $13^{3}/_{4}$’ + $1/2 \times 28^{31}/_{32}$’ = $42^{23}/_{32}$’).
4. An identically equal (3, 4) rectangle in central position. The horizontal side is given by the distance between the intersections of the diagonals of the squares ($1/2 \times 42^{23}/_{32}$’ + $1/2 \times 42^{23}/_{32}$’ – $13^{3}/_{4}$’ = $28^{31}/_{32}$’).

We may conclude from this that the position of the central columns 3 is final. However, the position of the columns 2 is not the ultimate one, as the distance from the edge of the *geison* to the axis of the far column 2 is $59^{15}/_{32}$ feet, that is $^{1}/_{32}$’ (about 1 cm) too long to represent the horizontal ‘perpendicular’ of the golden isosceles triangle. Thus the shifting of these columns is necessary for yielding an additional mathematical feature doubly (Fig. 4b). There is no doubt that the architect worked out all the elements of his design by calculation, e.g., GP + Tc + $7^{1}/_{2}$T + 8M – $^{1}/_{32}$’ gives the ‘perpendicular’ the correct length. But what about his intentions? It cannot be seen at a glance whether the vertex of the triangles has to be directed towards the interior of the temple or outwards. It is important to notice

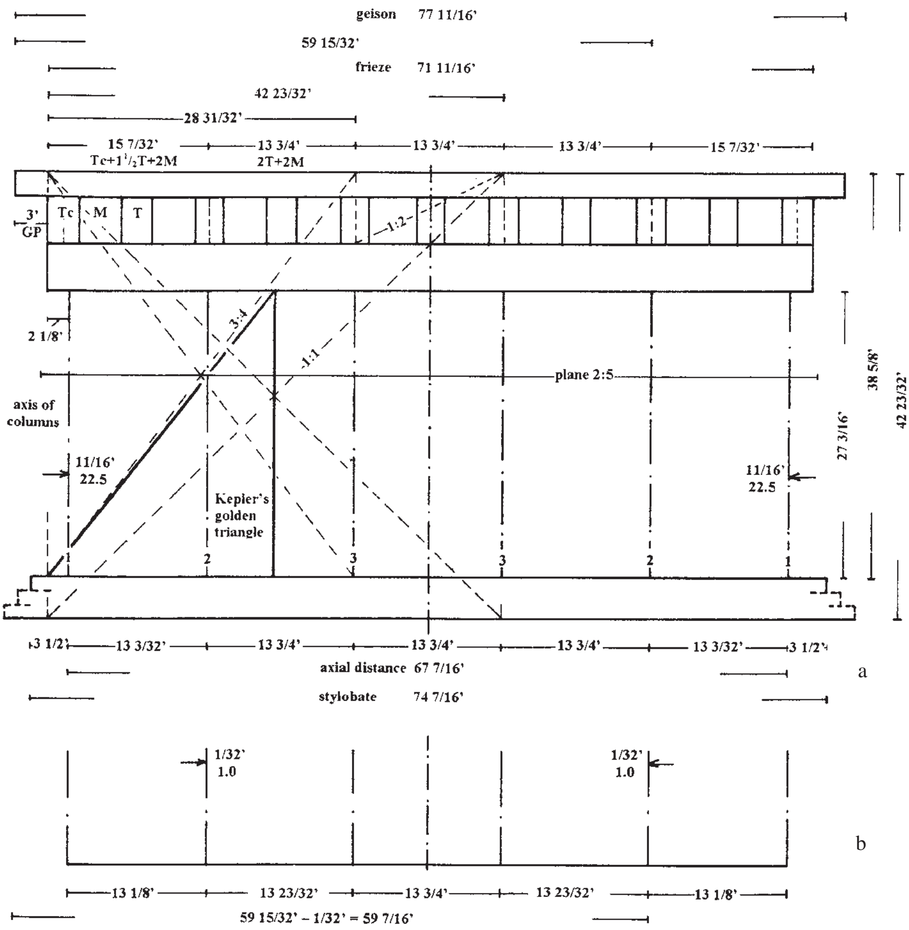

Fig. 4. The front of the Hera II temple at Paestum; 4a: The axis of columns 2 and 3 is in line with the centre of a triglyph while squares and Pythagorean (3, 4) rectangles are present; 4b: The 'perpendicular' ($59^{7}/_{16}$', in *horizontal* position) of the golden isosceles triangles comes out from shifting columns 2 (executed).

that the way of designing the length of the 'perpendicular' *fixes the desired position of the triangle with regard to the central square*. Thus it was not an adequate execution of the architect's intentions to uphold columns 2 in ideal aesthetic position and to obtain the length of the perpendicular by giving the *geison* projection $2^{31}/_{32}$ instead of 3 feet.

One diagonal of the square in Fig. 4a intersects the temple axis at the top of the

Table 3. The front of the second temple of Hera at Paestum			
First published by Krauss in 1941 (K) or by Mertens in 1984 (M)		Measured (cm)	Interpretation (1' = 32.66 cm)
Fig. 4a (only columns 1 shifted)			
Axial distance (AD)	M214	2203.3	2202.5 = $67^{7}/_{16}$
Stylobate length (SL) w/e	K46	2429.6/2431.6	2431.1 = $74^{7}/_{16}$
Frieze length (FL) = 2Tc+9T+10M	calc.	2338.8	2341.3 = $71^{11}/_{16}$
Tc+$1^{1}/_{2}$T+2M (see Table 2 for these parts)	calc.	496.5	497.0 = $15^{7}/_{32}$
Geison projection (GP)	K53	98.2	98.0 = 3
Geison length = FL + 2GP	calc.	2535.2	2537.3 = $77^{11}/_{16}$
Corner column axis to edge of architrave and frieze = 1/2(FL – AD)	calc.	67.8	69.4 = $2^{1}/_{8}$
Shifting of corner column = $2^{1}/_{8}$' – $^{1}/_{2}$Tc	calc.	20.9	22.5 = $^{11}/_{16}$
Axial intercolumn., at corner = $15^{7}/_{32}$' – $2^{1}/_{8}$'	calc.	428.7	427.6 = $13^{3}/_{32}$
Axial intercolumniation, normal = 2T + 2M	calc.	448.6	449.1 = $13^{3}/_{4}$
Corner column axis to edge of stylobate = 1/2(SL – AD)	calc.	113.2/114.2	114.3 = $3^{1}/_{2}$
Fig. 4b (final touch: shifting of columns 2)			
Axial intercolumniation, at corner	K46	430	428.7 = $13^{1}/_{8}$
Axial intercolumniation, intermediate	K46	447	448.1 = $13^{23}/_{32}$
Axial intercolumniation, at centre	K46	448	449.1 = $13^{3}/_{4}$

architrave because the sum of frieze height and *geison* height (see Table 2) is equal to half the axial intercolumniation $13^{3}/_{4}$'. It follows that the diagonals of both squares enclose around the temple axis a square whose diagonal is equal to the short side of the Pythagorean rectangle ($42^{23}/_{32}$' – 2 x $6^{7}/_{8}$' = $28^{31}/_{32}$').

The second temple of Hera embodies more mathematical theory. The so-called triangle of Kepler, the right-angled golden triangle, is also present. Fig. 4a shows the construction: drop a line from the bottom of the architrave to the stylobate in such a way that the line goes through the intersection of the diagonals of the square. Connect the starting-point of the line with the projection of the frieze edge on the stylobate. The vertical side of the triangle is equal to the column height $27^{3}/_{16}$'. The base is half the side of the square, that is $21^{23}/_{64}$ feet. The hypotenuse can be calculated: 34.5743.. feet. The ratio of base to hypotenuse is 0.61778.., which gives a vertical angle of amply 38°9' (the golden sine 0.6180 represents 38°10'). The decisive factor in the design of Kepler's triangle is the column height. It is worth noting that there is an alternative for the exposure of Kepler's triangle: drop the relevant line through the intersection of the diagonals of the Pythagorean (3, 4) rectangle and connect the starting-point with the temple axis at stylobate level. Obviously, the interpretation of the architect's intentions is not as easy as one might wish. In my opinion, however, this alternative cannot be inserted in the total plan.

The origin of Pythagorean mathematics

Really, this temple is a source of primary importance! The origin and spread of mathematics is still a mystery as historians of mathematics are unable to derive all aspects of Greek mathematics from a single origin (Seidenberg 1988, 101-104). Pythagorean number triples are known of about 1800-1600 BC in Babylonia (tablet Plimpton 322), of about 1800 BC in Egypt (Rhind mathematical papyrus) and of uncertain date in India (The Sulva-sutras, a collection of rules on altar constructions). There is no general agreement on the date of the Sulva-sutras. The suggested dating of the oldest rules ranges from 800 to 100 BC. As to number triples a date later than 600 BC is not really a problem as we have two alternatives for the origin of these triples. In 1962, however, Seidenberg (1962, 509-511) already pointed out that Greek mathematics has an Indian look because the gnomon was known to the Indians but not to the Babylonians and Egyptians. It follows that the gnomon might be of Indian origin, irrespective of the precise date of its invention. The spread of knowledge can be derived from the places where its practical execution can be established. Thus it is certainly of interest to observe that the square with side $42^{23}/_{32}$' consists of an L-shaped figure (gnomon) and a square with side $35^{27}/_{32}$' (half the frieze length). This way of building up squares by the adjunction of gnomons was known to Philolaos (Heath 1956, volume I, 351). On the other hand, Thales (about 585 BC) has never been credited with the gnomon. Thus, in spite of the difficulties in dating the Sulva-sutras on linguistic or literary grounds, a direct borrowing by the Greeks from India in the person of Pythagoras is plausible as the Hera temple II was designed some decades after his death. Of course, the alternative is the invention of the gnomon by a Pythagorean mathematician working before 460 BC[8]. Who borrowed from whom?

The executed front: observations

In the preceding section I have given a detailed description of the development of the design, all the while pointing to the large number of virtual figures the plan embodies. I will not present all these figures in a small scale drawing as the result is a disorderly heap. The reader who wishes to know all the ins and outs is well advised to make his own drawing on a large scale. The very essential information I give on a small scale (Fig. 5) suggests more than it really is: the illusion of a comprehensive geometrical concept. I resort to calculations to take the illusion away. Given: AB = $1/2 \times 42^{23}/_{32}' - (2^{1}/_{8}' + 13^{1}/_{8}') = 6^{7}/_{64}'$, BC = $27^{3}/_{16}' - 1/2 \times 38^{5}/_{8}'$

[8] Seidenbergs explanation (1962, 489) appears too much like special pleading when he says: "It is true that those who maintained the priority of Indian geometry may have claimed too much when they said that Greek geometry came from India: what they should have said was that Indian geometry and Greek geometry derive from a common source". This is a position difficult to defend as it always requires that the Greeks got in touch with the hypothetical common source of the gnomon between about 550 and 460 BC. Incidentally, Chinese mathematics (Jiu zhang suanshu or "Nine chapters of the mathematical art") is equally difficult to date (suggestions: 1030 BC, 221 BC, 100 AD). The gnomon seems to be used in the 9th chapter (Seidenberg 1988, 108-109).

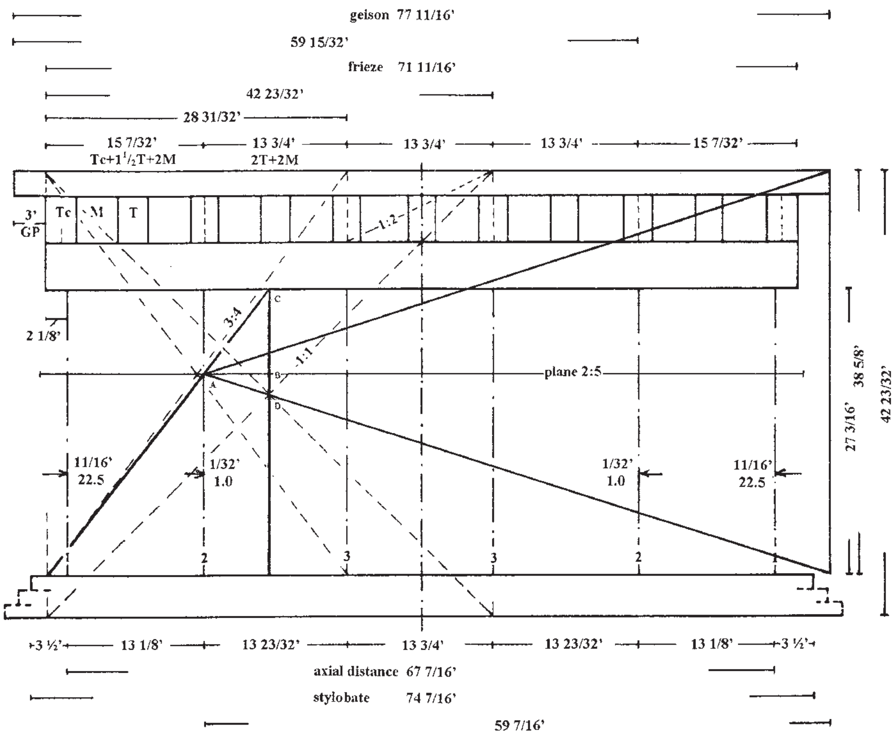

Fig. 5. The front of the second temple of Hera at Paestum as executed. The golden isosceles triangle has been put in correct position, but calculations refute the suggestion of a comprehensive geometrical concept.

$= 7^{7}/_{8}$’ and BD = $1/2(42^{23}/_{32}$’ – $38^{5}/_{8}$’) = $2^{3}/_{64}$’ or half the platform height $4^{3}/_{32}$’ (see Table 2).

Tangent angle ACB = AB:BC = 0.77579, which belongs to an angle of 37°48’ and so differs too much (about 21 minutes) from the vertical angle of Kepler’s triangle calculated above. It follows that the hypotenuse of the golden right-angled triangle does not pass through the vertex A of the golden isosceles triangle.

If one leg of the golden isosceles triangle passes through D, angle BAD has to be 18° (see Fig. 3; tangent 18° = 309:951 = 0.3249). Instead of this we find tangent BAD = BD:AB = 0.3350 (about 18°31’). These results make it unlikely that the architect’s plan originates from only one single coherent geometrical construction. I defend the position that the plan is a composition of various figures.

The executed front: interpretation

The axis of the temple is axis of symmetry. The architect started the execution of his design with a square of which the side is $35^{27}/_{32}$ feet, that is in horizontal

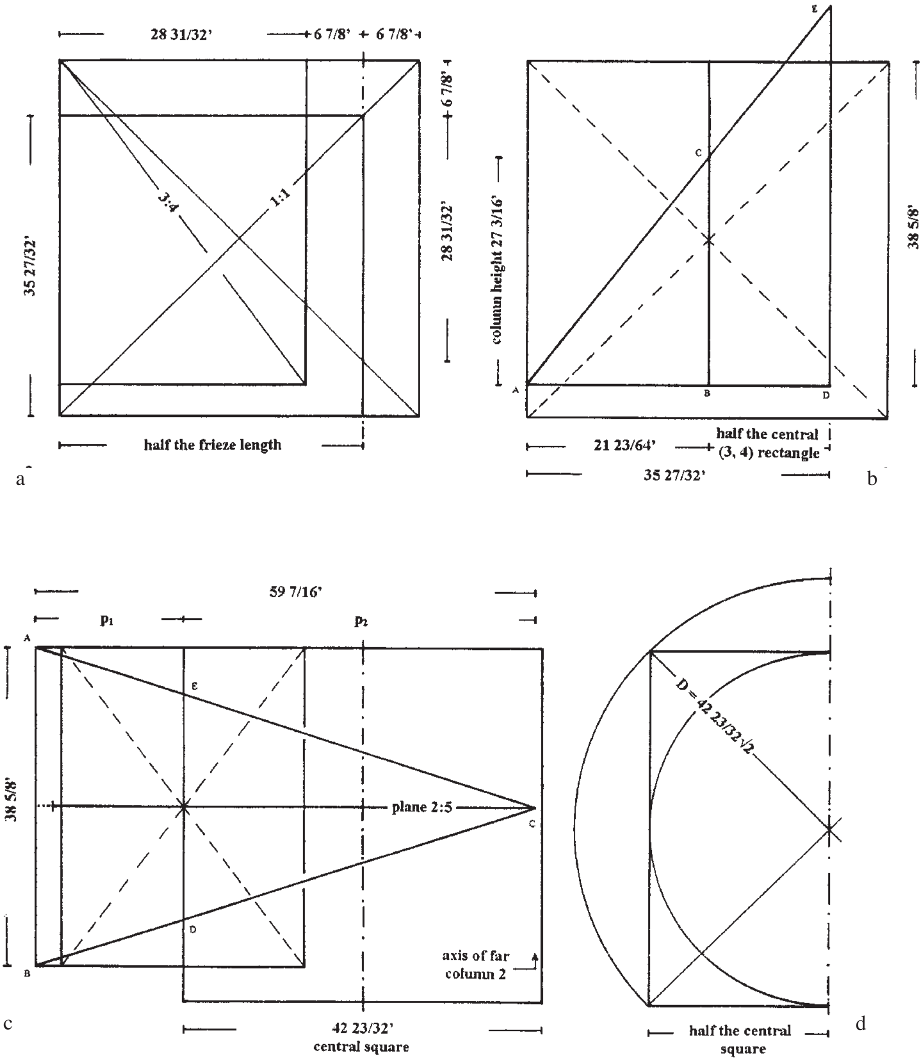

Fig. 6. Virtual figures connected by the number 5 on the front of the Hera II temple at Paestum; 6a: Two diagonals are in harmony with the diagonal of the Pythagorean (3, 4) rectangle; 6b: The hypotenuse of the right-angled golden triangle ADE is in harmony with the diagonal of the (3, 4) rectangle; 6c: Halving the area of the golden isosceles triangle by line DE; 6d: The areas of the circles are in harmony with the area of the golden isosceles triangle.

direction half the frieze length and in vertical direction the distance from the bottom of the platform to the upper side of the architrave (Fig. 6a). This side cannot have any length as the architect aimed at the base of the golden isosceles triangle expressible as telling number: $35^{27}/_{32}$' – $6^{7}/_{8}$' = $28^{31}/_{32}$ feet; $28^{31}/_{32}$' x 4/3 = $38^{5}/_{8}$ feet, that is the telling number 618 (dactyls). The upright base $38^{5}/_{8}$' was placed at right angles to the edge of the horizontal *geison* and the same value represents the long side of the Pythagorean (3, 4) rectangle at the edge of the frieze (Fig. 6c). I discuss only the left side of the front, but always indicating the axis of symmetry for easy reference.

But let me first explain the architect's design in a nutshell. The front of the temple has two corners where identical virtual figures exist, but the Pythagorean harmonic concept appears around the temple axis applying to only one figure. The central (3, 4) rectangle (Fig. 6b), the central square with side $42^{23}/_{32}$' (Fig. 6c) and the small central square with upright diagonal $28^{31}/_{32}$' – whose sides are enclosed by the diagonals of the squares in the corners (Fig. 6a) – have already been discussed above. The area of the isosceles triangle emerges in central position by halving of the area of the golden isosceles triangles in the corners (Fig. 6c). Therefore I add two circles to the great central square, because the areas (see *infra*) are significantly related to the area of the golden isosceles triangle (Fig. 6d). Finally, the odd man out among the figures is that of the golden right-angled triangle. This triangle (as well as his counterpart on the right) has been enlarged by lengthening the hypotenuse AC up to the temple axis in E (Fig. 6b).

In talking about the circles I hinted at connections between the constructed shapes. Indeed, the architect did not simply design a nice arrangement of figures in one picture.

If my interpretation is right, the ideas the architect worked with seem to have been based exclusively on ratios. The architect's idea produced a system with carefully calculated ratios in inner harmony with each other, as they are all related to the number 5. No wonder that the diagonal 5 of the Pythagorean (3, 4) rectangle is related to three other straight lines. One of these lines is the diagonal of the great square. To us the diagonal of a square is $\sqrt{2}$ times the side, but the Pythagoreans replaced the square root of 2 by a rational approximation. However, the side $42^{23}/_{32}$', which is 1367 in number (half dactyls), has no integral counterpart for the diagonal. The necessity of finding a rational expression for $\sqrt{2}$ arose from this problem. It follows that we go to suggestions concerning the applied rational approximation (Table 4).

The very accurate rational approximation 577/408 for $\sqrt{2}$ is used in the Indian Sulva-sutras, but expressed differently: the diagonal of a square is 1 + 1/3 + 1/(3x4) – 1/(3x4x34) of a side (Seidenberg 1962, 515). As said above, the architect of the first temple of Hera at Paestum used 437/309. However, he did not use it as a makeshift but as the obvious means to carry out his plan, that is the construction of a square of which the diagonal is (to us: practically) rational (437 double dactyls) along with one side also as base of the golden isosceles triangle expressed as telling number (618 dactyls). Haselberger discovered that 99/70 (in

Table 4. The ratio of the diagonal $48^9/_{32}$’ of the (3, 4) rectangle to the diagonal of the square with side $42^{23}/_{32}$’		
Factor	Diagonal of the square = factor x $42^{23}/_{32}$’	The ratio is almost 4:5
17/12 = 1.416666r	60.5182..’	$48^9/_{32}$:60.5182.. = 3.98898..:5
99/70 = 1.414285..	60.4165..’	$48^9/_{32}$:60.4165.. = 3.99569..:5
437/309 = 1.414239..	60.4145..’	$48^9/_{32}$:60.4145.. = 3.99583..:5
577/408 = 1.414215..	60.4135..’	$48^9/_{32}$:60.4135.. = 3.99589..:5
√2 = 1.414213..	60.4134..’	$48^9/_{32}$:60.4134.. = 3.99590..:5
123/87 = 1.413793..	60.3954..’	$48^9/_{32}$:60.3954.. = 3.99709..:5
89/63 = 1.412698..	60.3487..’	$48^9/_{32}$:60.3487.. = 4.00018..:5
7/5 = 1.4	59.80625’	$48^9/_{32}$:59.80625 = 4.03647..:5

dactyls) was used in a drawing at Didyma (*ca*. 250 BC) (Haselberger 1980, 240 and fig. 1). Again at Didyma we find 123/87, that is 3/3 x 41/29, used in a drawing which depicts the construction of a recessed panel in a ceiling (coffer)[9]. I discovered the theoretical value of 89/63, but the question is whether the ancients knew it. According to Wells the Pythagoreans used 7/5, but he gives no reference to check his statement (Wells 1986, s.n. 1,414). In my opinion, however, 7/5 gives no acceptable result. The approximation 17/12 is indeed a nice simple one and of practical use. I take the view that the architect might have adopted 17/12 as we find these numbers used in halving the area of the golden isosceles triangle (below).

The second line related in harmony to the diagonal $48^9/_{32}$’ (3:5) is of course the diagonal of the small central square (Fig. 6a) as the latter is as long as the short side of the (3, 4) rectangle.
The third line is the hypotenuse of the enlarged right-angled golden triangle ADE (Fig. 6b). DE is unknown, but the triangles ABC and ADE are congruent. So it is easy to calculate DE from AD/AB x BC, which gives 45.62408.., rounded off 45.625 = $45^5/_8$’. DE is the sum of $38^5/_8$ and 7 feet. Unfortunately, there is no measurement available above the horizontal *geison*. Most probably these 7 feet (= 228.6 cm) represent the height of the *tympanon*, which is still extant on the east-side of the temple[10]. Finally, the theorem of Pythagoras gives the length of the hypotenuse AE = 58.0208.. feet. The ratio of this hypotenuse to the diagonal $48^9/_{32}$’ is 6.008.. : 5 or almost 6:5.

[9] Haselberger 1983, 96. The architect planned a square coffer with side equal to half the axial intercolumniation $17^3/_4$’, but a square with side $8^7/_8$’ or 71 double dactyls cannot easily be constructed as it lacks a diagonal in whole double dactyls. Such a problem is easily solved by using a suitable larger square adding lines parallel to the sides on the inside at a distance of a dactyl or a double dactyl. This is exactly what Haselberger found within a square with side length 325 cm: 324.7 cm = $10^7/_8$ feet = 87 double dactyls; the diagonal of this square is (almost) 123 double dactyls.

[10] See photo 42 in Krauss 1976. The height of the *tympanon* in the temple at Segesta is 231.4 cm (Mertens 1984, 211).

More on the golden isosceles triangle

Let us next discuss halving the golden isosceles triangle ABC (Fig. 6c). In a triangle, a line parallel to the base AB divides the other sides proportionally. The same applies to other lines starting from the vertex C, e.g. the perpendicular. A modern expression for halving the area states that the ratio has to be $1:\sqrt{2} + 1 = p_1:p_2$. For the time being, I follow Wells (1986), and put $\sqrt{2}$ at 7/5, which gives $p_1:p_2 = 1:2.4$ and, indeed, a division of the area into halves according to the architect's intention. But I do not believe that the Pythagoreans knew enough algebra to formulate such a ratio wherein the value of $\sqrt{2}$ is decisive, but used it without the presence of a square. As a matter of fact, there is no need for supposing such knowledge as the question can be solved directly, without reference to the modern expression of the ratio, because 1:2.4 = 5:12. In words: divide the perpendicular of any triangle into 17 parts and assign 5 parts to the trapezium (ABDE) for halving the triangle[11]. Such a formulation can easily be attributed to the Pythagoreans as the ratio is stated in whole numbers and gives a fair approximation to halving the area. Furthermore, it follows that the Pythagoreans knew that the area of a trapezium is computed as half the sum of the parallel sides times the width.

Part p_1 of perpendicular $59^7/_{16}$' can be found in Fig. 5: the sum of the *geison* projection (3') and half the short side of the (3, 4) rectangle $28^{31}/_{32}$' is $17^{31}/_{64}$ feet. The other part p_2 is 12/5 p_1 = 41.9625, which is close to 41.953125 = $41^{61}/_{64}$ feet or $59^7/_{16}$' – $17^{31}/_{64}$'. Half the area of the golden isosceles triangle ABC is 573.9.. sq. feet. Line DE, a part of the left vertical side of the great central square, divides the triangle into a trapezium and a triangle; its length is 12/17 AB = 27.26.. feet. The area of the separate figures deviate less than 0.4% from half the area triangle ABC. The area of trapezium ABDE is 1/2(AB + DE) x p_1 = 576.0.. sq. feet and the area of triangle CDE is 1/2 DE x p_2 = 571.9.. sq. feet. The architect was satisfied with this result as the ratio of area trapezium to area triangle = 1.007..:1 or almost 1:1. The case of the golden isosceles triangles is decisive proof for considering the constructions in the corners of the temple as formative. We focus therefore our attention on the temple axis.

The area of the golden isosceles triangle, 1/2 x $38^5/_8$' x $59^7/_{16}$' = 1147.8.. sq. feet, is in harmony with the area of the inscribed and circumscribed circle of the central square with side $42^{23}/_{32}$ feet (Fig. 6d). To the purpose of calculation I use for π the Archimedean value 22/7[12].

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[11] In Babylonia and Egypt the mathematical importance of a problem lies in its arithmetical solution. Not surprisingly, the examples given in the texts are so arranged that the calculation is an easy exercise. A Pythagorean example for halving a triangle might have base 153 and perpendicular 238 as the calculator does not find broken numbers on his way. Moreover, if the triangle is taken as isosceles the side is (almost) a whole number.

[12] The rational approximation 99/70 for $\sqrt{2}$ is of special interest if Greek mathematicians working in the Hellenic period knew that $\sqrt{2}$ and π are about in the ratio of 9:20, as 22/7 x 9/20 = 99/70.

The area of the inscribed circle is 1/4 x 22/7 x D^2 = 1433.8.. sq. feet. The ratio of area triangle to area circle is 4.002.. : 5 or almost 4:5.
The areas of two circles are to each other as the squares on their diameters. The diameter of the circumscribed circle is $42^{23}/_{32}\sqrt{2}$ feet. It follows, that the area of the circumscribed circle is two times the area of the inscribed circle. So the ratio of area triangle to area of the former circle is almost 2:5.

One system of harmony
The result so far gives an impression of two separate systems of harmony. In fact, however, we are dealing with just one, as the systems of line and area are connected. The ratio of the diagonal $48^{9}/_{32}$' of the (3, 4) rectangle to the diameter of the circumscribed circle $42^{23}/_{32}\sqrt{2}$' is 3.9959.. : 5 or almost 4:5. However, I replace this by saying that the ratio of the diagonal to the radius of the circumscribed circle is almost 8:5. I admit, there is no point in doing this unless the still missing ratio 7:5 turns up (see *infra*).

Let us now return to the free floating rectangular plane for a discussion of the length of the side on the front of the temple (Fig. 1). Following Krauss, I accepted its length as equal to half the sum of the frieze and stylobate lengths: $1/2(71^{11}/_{16}' + 74^{7}/_{16}') = 73^{1}/_{16}$ feet. As an aside I stated above that the length was fixed this way to be able to develop the design. Thus our attention now shifts to the intervening stages of this part of the plan. At first I wondered at its length as the perpendicular of the golden isosceles triangle exceeds the limit of the floating plane (Fig. 6c). Gradually it became clear that the architect selected this length as a compromise to kill two birds with one stone, that is, a plane in the ratio 2:5 and in the short side of the plane the embodiment of the ratio 7:5. Probably the architect was a Pythagorean mathematician, who not only used measures with precision, but did so in a way that was explicitly mathematical. He started off by defining the width of metope M and normal triglyph T, which, among other things, resulted into a measure of $52^{3}/_{16}$ feet (Fig. 1), that is the sum total of 8 M and 7 T, centred around the temple axis and equal to twice the hexagon's apothem (see *infra*). After finishing the frieze ($71^{11}/_{16}$'), the side of the floating plane was fixed as 7/5 x $52^{3}/_{16}' = 73^{1}/_{16}$ feet and finally the stylobate resulted from two times $73^{1}/_{16}$' less $71^{11}/_{16}' = 74^{7}/_{16}$ feet. It took me a long time before I understood the order in which the events follow each other, as it cannot be grasped before the total design is clear.

Let me explain this. The architect was faced with the problem that there is a limit to everything. He was unable to express the radius of the circumscribed circle of the square directly in a dimension of the temple. I suggest that the architect worked on the premise that the status of any measure can be formally acknowledged either by measurement or by a sound line of reasoning. As an example of the latter I may refer to the argumentation to find the sides of the floating plane. Thus it is not enough to argue that the area of the circumscribed circle is two times the area of the inscribed circle, as we cannot be certain that the architect incorporated the circumscribed circle in his plan. Up till now it is only known that it can be done within a mathematical design based on number 5 as its diameter is properly related to the diagonal of the (3, 4) rectangle by the approximate ratio of 5 to 8. However, instead of circumstantial evidence, explicit evidence is needed before the circumscribed circle can be accepted with full confidence as part of the plan.

A virtual hexagon

The architect introduced the circumscribed circle ingeniously, adding splendour to the design. The length of the side of a polygon within a circle is fixed by the radius of the circle and the apothem. Thus there are three unknowns. The polygon that offers a way out of the difficulty is the hexagon as the side is equal to the radius of the circle. Thus it is no surprise that the architect gives the apothem and points to it by the ratio 7:5. Our observation gives reason of existence to the circumscribed circle of the square as part of the design.

The construction of the virtual hexagon is cleverly planned and skilfully made (Fig. 7), but what about its accuracy? A few calculations answer this question. First the apothem (A) using the identity $A^2 = R^2 – (1/2\ R)^2$. The apothem of a hexagon within a circle with radius 30.2067.. feet is 26.1597.. feet. The architect's approximation for the apothem $26^3/_{32}$' is 1/4 % too short.

Given the apothem $26^3/_{32}$', the two sides in vertical position work out 30.4343.. feet, that is 3/4 % too long. The other sides are 1/4(outline of hexagon 6R less 2 x 30.4343..') = 30.0929.. feet, that is 3/8 % too short. This result is accurate enough to accept the intention of the architect to produce a regular form of an especially desirable kind, a hexagon[13]. The result is approximating as it was done by the ratio of whole numbers, a method which is typical of the Pythagorean mathematicians. Certainly, the architect possessed no little mathematical knowledge. Historians of mathematics may decide whether this investigation of the principles of the second temple of Hera at Paestum enlightens the state of mathematics at his time.

The resulting harmonic composition

The final result of the architect's design, executed on the front of the temple is a harmonic composite in central position (Fig. 8) and its constituent parts are all connected by the number 5. But it is surprising that the built-in geometrical figures are virtual and so invisible for human beings. In other words, the usual visitors of the temple were not the target group the architect had in mind. It follows, that the architect aimed his design at superior beings that surpass men in qualities. I surmise that the structure was designed and executed for the benefit of the Pythagorean community, but was it intended to protect from harm by demons or to do honour to the gods? I leave it to the reader to answer this question.

[13] If regularity of the hexagon is accepted, the distance 8M + 7T = $52^3/_{16}$' represents the side of the equilateral triangle. The side is 1/4 % too short as 30.2067.. x $\sqrt{3}$ = 52.3195.. feet.

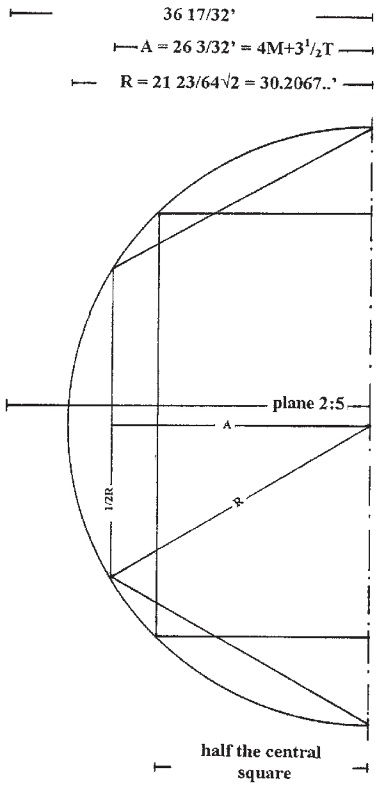

Fig. 7. The front of the Hera II temple at Paestum. Approximate construction of the side of a regular hexagon by the ratio $36^{17}/_{32}$' : $26^{3}/_{32}$' = 7:5 (see Fig. 1 for position of A).

The rest of this paper will be limited to a short discussion of the flank of the temple, not just with the intention of explaining the development of the long side of the temple from the front dimensions but also of demonstrating the results which can be achieved in the field of mathematics if a large series of columns is systematically shifted out of its ideal aesthetic position. The latter is partly hypothetical but of interest as this phenomenon has also been observed on Sicily, namely on the great temple of Segesta and the temple of Apollo at Syracuse. Moreover, the variants I present for the shifting, on basis of Krauss' concise description (below), causes one to see the clash between contradictory wishes. It is shown that advanced mathematics is realized at the cost of aesthetic qualities. In my opinion, aesthetic qualities recede for the benefit of performing Pythagorean mathematical concepts.

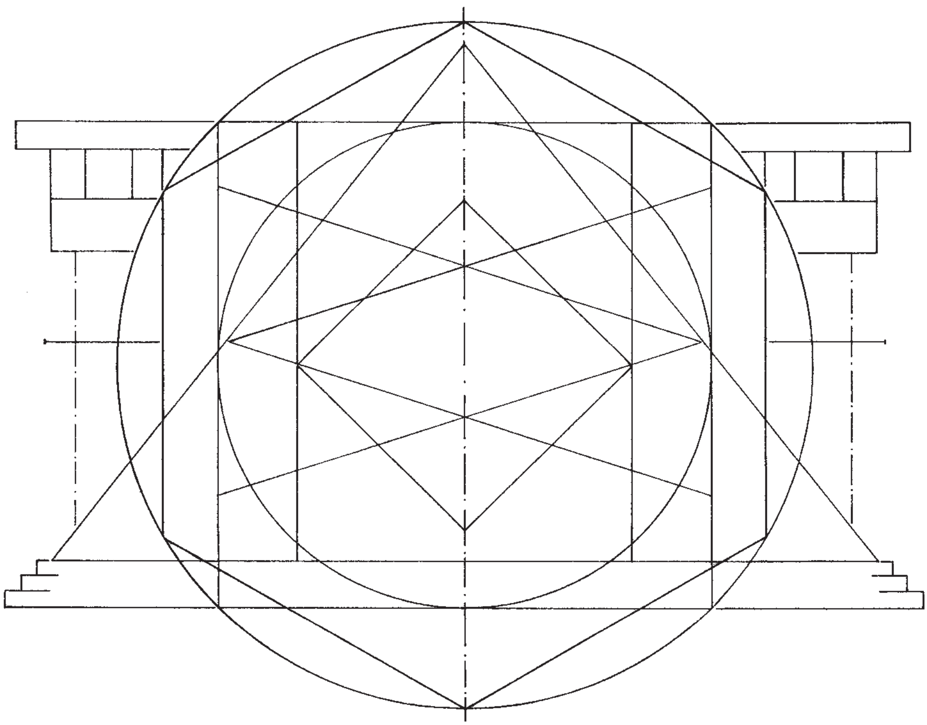

Fig. 8. The second temple of Hera at Paestum. Virtual harmonic composite at the axis of the front.

From front to flank design

The preliminary design of the front (Fig. 4a) has 6 columns, single angle contraction and 3 normal axial intercolumniations of $13^3/_4$ feet. On the flank (Fig. 9) we find at its preliminary stage 14 columns, double angle contraction and 9 uniform axial intercolumniations of $13^3/_4$ feet, as shown in the frieze. *Geison* and frieze become 110 feet (8 x $13^3/_4$') longer than on the front. To provide for double angle contraction the corner column and the column next to this column shift $^7/_{16}$ feet inwards. So the axial distance becomes $67^7/_{16}$' + 110' – $^7/_8$' = $176^9/_{16}$ feet and the axial spacing at the corners is as on front, that is $13^3/_{32}$ feet. The intermediate spacing becomes $13^3/_4$' – $^7/_{16}$' = $13^5/_{16}$ feet.

The axial distance, given by Mertens (Table 5), confirms the outline of this preliminary design of the flank as $176^9/_{16}$' = 2 x $13^3/_{32}$' + 2 x $13^5/_{16}$' + 9 x $13^3/_4$'. Normally the stylobate length should become $74^7/_{16}$' + 110' – $^7/_8$' = $183^9/_{16}$ feet, but it may be remembered that the long side of the virtual horizontal plane (5/2 x $73^1/_{16}$' = $182^{21}/_{32}$ feet) has to be half the sum of frieze and stylobate lengths. The

Table 5. The flank of the second temple of Hera at Paestum			
First published by Krauss in 1941 (K) or by Mertens in 1984 (M)		Measured (cm)	Interpretation (1' = 32.66 cm)
Axial distance (AD)	M214	5766.3	5766.5 = $176^{9}/_{16}$
Stylobate length (SL) south/north	K46	5989.1/5996.0	5997.2 = $183^{5}/_{8}$
Frieze length (FL) = 2Tc+25T+26M	calc.	5927.6	5933.9 = $181^{11}/_{16}$
Geison projection (GP)	K53	98.2	98.0 = 3
Geison length = FL + 2GP	calc.	6124.0	6129.9 = $187^{11}/_{16}$
Corner column axis to edge of architrave and frieze = 1/2(FL – AD)	calc.	80.7	83.7 = $2^{9}/_{16}$
Shifting of corner column = $2^{9}/_{16}$' – $^{1}/_{2}$Tc	calc.	33.8	36.7 = $1^{1}/_{8}$
Corner column axis to edge of stylobate; remark: 1/2(SL *on north side* – AD) = 114.9	K46	114.7/115.2	115.3 = $3^{17}/_{32}$
Axial intercolumniation, at corner	K46	426	425.6 = $13^{1}/_{32}$
Axial intercolumniation, intermediate	K46	434.8	434.8 = $13^{5}/_{16}$
Axial intercolumniation, normal (9); remark: AD – 2(426 + 434.8) = 9 x 449.4	K46	450.2	not normal but variable

requisite length of the stylobate $183^{5}/_{8}$' was achieved by putting the axis of the corner columns at a distance of $3^{17}/_{32}$ feet off the stylobate edge, instead of $3^{1}/_{2}$ feet as on front. Aesthetically, the preliminary design is excellent as the triglyphs of 10 columns are centred over the column axis. However, a glance at Table 5 shows that the intermediate axial intercolumniation is correct while the one at the corner is smaller than predicted. This is an indication that the architect desired eagerly to seek some mathematical aim at the flank too. Certainly, such a design requires the shifting of more columns. Krauss only makes in passing a remark on shifting of the columns on the flank (Krauss 1976, 53). Unfortunately, his remark is tantalizing in its vagueness, but still one may hope to glean some information from it of the architect's intention. So much is clear that many (or all) columns are shifted and that the direction of shifting is not uniformly inwards or outwards. That will do to frame several interesting hypotheses.

Setting up on the stylobate a plane in the ratio 2:5 that intersects the horizontal plane in the same ratio seems a nice idea to realize for an architect occupying himself in the number 5 (Fig. 9a). The vertical measure $38^{5}/_{8}$' is known to us on the front as long side of the (3, 4) rectangle and as base of the golden isosceles triangle. The shifting of both columns 4 $^{5}/_{32}$ feet outwards results into the horizontal side $96^{9}/_{16}$'. It is to be expected that the architect did not neglect the aesthetic side of his plan. Changing only the position of columns 4 would cause a harsh contrast between the third and fourth axial intercolumniation (third: $13^{3}/_{4}$' – $^{5}/_{32}$' = $13^{19}/_{32}$' = 444.0 cm; fourth: $13^{3}/_{4}$' + $^{5}/_{32}$' = $13^{29}/_{32}$' = 454.2 cm). Therefore he strove to obtain a gradual change. Outside the plane, the first and third axial intercolumniation were modified (first intercolumniation: $13^{3}/_{32}$' – $^{1}/_{16}$' = $13^{1}/_{32}$', indicated in table 5 as factually established; third intercolumniation: $13^{19}/_{32}$' + $^{1}/_{16}$' = $13^{21}/_{32}$' and so compensating for the modification of the first). This is the design shown in Fig. 9a. Systematic shifting of columns 5, 6 and 7 can be realized in several ways. It all

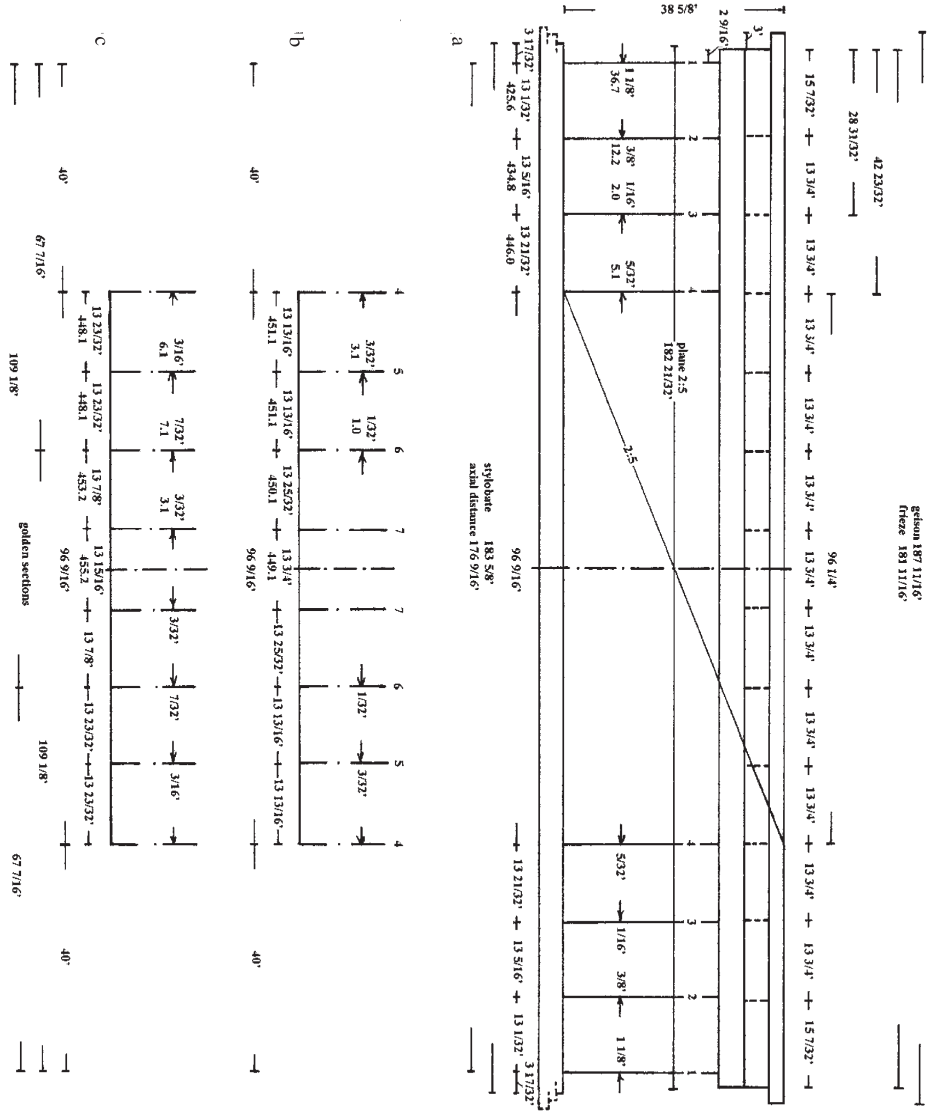

Fig. 9. The flank of the Hera II temple at Paestum; 9a: Double angle contraction after Krauss and central part interpreted by author. Preliminary design a can be executed in a meaningful way either as b or c; 9b: Aesthetic solution (central columns 7 not shifted), comparable with the executed plan of the front; 9c: Advanced mathematical design, in line with the result on the front.

depends on the complexity of the design. In Fig. 9b the central columns 7 continue to be centred as on the front and the shifting of columns 5 and 6 is not too obvious. Fig. 9c depicts a solution that is also in line with the results on the front. This idea is based on the observation that the axial distance on the flank ($176^{9}/_{16}$') is related to the axial distance on the front ($67^{7}/_{16}$') as 2.618..:1. Anyone who is familiar with geometrical progressions will immediately recognize this ratio as the square of the golden section 1.618..:1. But let me illustrate its origin with the well-known series of Fibonacci (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...): 144 divided by 55 gives 2.618…and 89 divided by 55 results into 1.618.., which is an approximation of the golden section. So the division of the axial distance on the flank $176^{9}/_{16}$' into $67^{7}/_{16}$' and $109^{1}/_{8}$' represents the golden section in practical form like the golden triangles on the front. According to modern standards, however, the measure of column shifting is less satisfying than in Fig. 9b. However, such a conclusion has to be of course based on measurement. Nobody is looking at a frieze situated about 12 m above floor level, separated from the columns including abacus by an architrave $1^{1}/_{2}$ m in height, trying to settle its aesthetic qualities by assessing the distance in cm between the virtual axis of each column and the virtual axis of its triglyph. Instead, the onlooker is giving his whole mind to the general impression of the temple.

Conclusion

The virtual harmonic composite of various geometrical figures related to each other by the number 5 is convincing evidence that Pythagorean mathematics is flourishing at Paestum about 460 BC. The purpose of this construction is open to discussion. The discovery of a virtual horizontal plane that nowhere touches the building, clears a way for a change in studying Greek temple architecture. I have tried to illuminate the practical feasibility and flexibility of methodical shifting of columns. The results on the front can be accepted with full confidence as the available measurement is not too scanty. The argument outlined above for the last alternative on the flank, however plausible, cannot be regarded as conclusive. What is really needed is a full account on actual executed shifting of the columns.

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